Dynamical Critical Properties of the Random Transverse-Field Ising Spin Chain

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We study the dynamical properties of the random transverse-field Ising chain at criticality using a mapping to free fermions, with which we can obtain numerically exact results for system sizes, L, as large as 256. The probability distribution of the local imaginary time correlation function $S(\tau)$ is investigated and found to be simply a function of $\alpha \equiv -\log S(\tau)/\log\tau$. This scaling behavior implies that the *typical* correlation function decays algebraically, $S_{\rm typ}(\tau) \sim \tau^{-\alpha_{\rm typ}}$, where the exponent $\alpha_{\rm typ}$ is determined from $P(\alpha)$, the distribution of α. The precise value for $\alpha_{\rm typ}$ depends on exactly how the "typical" correlation function is defined. The form of $P(\alpha)$ for small α gives a contribution to the *average* correlation function, $S_{\rm av}(\tau)$, namely $S_{\rm av}(\tau) \sim (\log\tau)^{-2x_m}$, where x_m is the bulk magnetization exponent, which was obtained recently in Europhys. Lett. **39**, 135 (1997). These results represent a type of "multiscaling" different from the well-known "multifractal" behavior.

I. INTRODUCTION

Quantum phase transitions show a number of remarkable features and have attracted considerable interest in recent years. They occur at zero temperature and are driven by quantum, rather than thermal, fluctuations. Hence, they are induced by varying a parameter other than the temperature, such as an applied transverse magnetic field. In particular, systems with quenched disorder show surprising properties near a quantum critical point. For instance, it was found that in quantum Ising spin glasses[1,2] and random transverse-field Ising ferromagnets,[3,4] as well as in "Bose glass" systems (which have a continuous symmetry of the order parameter but lack "particle-hole" symmetry[5]), all or part of the disordered phase shows features which are usually characteristic of a critical point. More precisely, correlations in time decay algebraically and thus various susceptibilities may actually diverge. This behavior is due to Griffiths-McCoy singularities[6,7], which arise from rare clusters which are more strongly coupled than the average, and it has become clear that their effect is much more pronounced near a quantum transition than near a classical critical point.

One of the simplest models exhibiting the characteristic features of a quantum phase transition is the random transverse-field Ising chain, defined by the Hamiltonian

$$H = -\sum_{i=1}^{L-1} J_i \sigma_i^z \sigma_{i+1}^z - \sum_{i=1}^{L} h_i \sigma_i^x, \qquad (1)$$

where the $\{\sigma_i^\alpha\}$ are Pauli spin matrices at site i and the interactions J_i and the transverse fields h_i are random variables with distributions $\pi(J)$ and $\rho(h)$, respectively.

A lot of results on the critical and off-critical properties of this model have been obtained, both analytically and numerically. The ground state properties of the Hamiltonian in Eq. (1) are closely related to a two-dimensional classical Ising model where the disorder is perfectly correlated along one direction, the latter model first being studied by McCoy and Wu[8]. Subsequently, the quantum model was studied by Shankar and Murthy[9], and recently the critical properties have been worked out in great detail by D. S. Fisher, using a real space renormalization group approach[3,10]. The quantum model, Eq. (1), has also been investigated numerically[10–14] using a mapping to free fermions by means of a Jordan-Wigner transformation.

We now briefly summarize some of the surprising features of this model. Distributions of *equal time* correlation functions are found to be very broad which leads to[3]: (i) different critical exponents for the divergence of the "typical"[15] and average correlation lengths, and (ii) the typical equal time correlation function at criticality falls off as a stretched exponential function of distance, quite different from the power law variation of the average. Recently, it has been shown that these main results are not restricted to one dimension, but also seem to hold in the two-dimensional random Ising ferromagnet[16,17].

A number of results for dynamics have also been found. For example, at the critical point the dynamical exponent z is infinite[3]. *Away from the critical point* the distribution of local (imaginary) time dependent correlation functions is very broad[12]. The average varies as a (continuously varying) power of imaginary time τ involving an exponent[18] $z'(\delta)$, where δ is the deviation from criticality. By contrast, the typical correlation function varies as a stretched exponential function of time. Quite detailed information on the whole distribution of time dependent correlation functions in the paramagnetic phase has also been found[12]. *At the critical point* the average correlation function is found to decay with an inverse power of

the log of the time[13], corresponding to the result $z = \infty$ mentioned above.

However, the *distribution* of time dependent correlation functions at the critical point has not yet been determined and is the focus of this study. We find that the distribution is very broad and can be expressed in terms of a single (logarithmic) scaling variable, defined in Eq. (6) below. This result implies that there is a continuous range of exponents α, somewhat analogous to (but also with important differences from) "multifractal" behavior[19,20].

We present our results in the following section, and we conclude with a discussion in Sec. III.

II. RESULTS

Throughout the paper we assume the following rectangular distributions for the couplings J_i and the transverse fields h_i

$$\pi(J) = \begin{cases} 1 & \text{for} \quad 0 < J < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\rho(h) = \begin{cases} h_0^{-1} & \text{for} \quad 0 < h < h_0 \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

which are characterized by a single control parameter h_0. The system possesses a critical point at $\delta = [\ln J]_{\rm av} - [\ln h]_{\rm av} = 0$, i.e. $h_0 = 1$, at which the distributions of the bonds and fields are equal. The lattice size is L and we impose *free* rather than the more conventional periodic boundary conditions[11,12].

For the numerical work we make use of the mapping of Hamiltonian (1) onto a model of free fermions[21–23]. Since the transformation has been used in previous work, we only give a brief summary here and refer to Refs.[11,12,14] for further details. For free boundary conditions, which we shall assume here, the most convenient representation is given in Refs.[11,24], necessitating only the diagonalization of a $2L \times 2L$ real, *tridiagonal* matrix. The spin operators occurring in the expectation value of the correlation functions can then be expressed as a product of fermion operators, which is evaluated using Wick's theorem. The resulting Pfaffian is then given by the square root of the determinant of a matrix, where the matrix elements can be calculated from the eigenvectors and eigenvalues of the Hamiltonian in the free fermion representation. The imaginary time correlation functions are always positive, so there is no ambiguity in sign when taking the square root. For $L \leq 128$ we average over 30000 realizations of the disorder, while for the largest size, $L = 256$, we average over 10000 realizations.

We calculate the probability distribution $P(\ln S(\tau))$ of the single site imaginary time correlation function

$$S_{ii}(\tau) = \langle \sigma_i^z(\tau) \sigma_i^z(0) \rangle \tag{4}$$

at the critical point, i.e. $\delta = 0$. For convenience, we will denote $S_{ii}(\tau)$ by $S(\tau)$ from now on. To obtain better

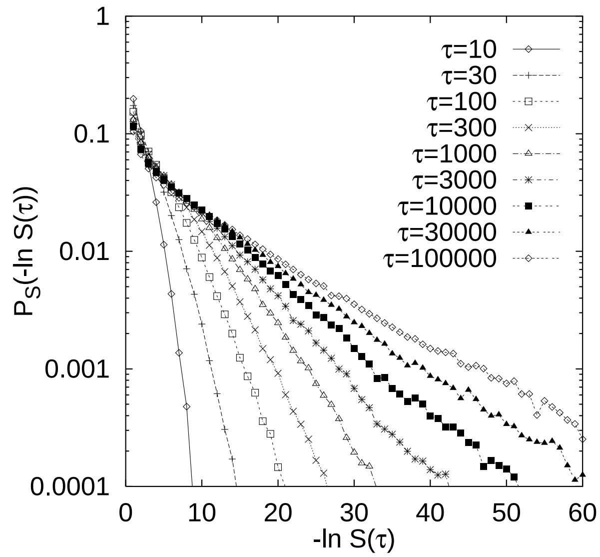

FIG. 1. The probability distribution, P_S, of $-\ln S(\tau)$ for $L = 128$ and different values of the imaginary time τ at the critical point $\delta = 0$. The data is averaged over 30000 samples.

statistics, we determine the correlation function for every second site from $L/4$ to $L/2$, making a total of $L/8$ sites. All sites are far from the boundary so we do not expect the results to be affected by boundary effects. The average correlation function is then given by

$$S_{\rm av}(\tau) = \frac{8}{L} \sum_i [S_{ii}(\tau)]_{\rm av}, \tag{5}$$

where $[\cdots]_{\rm av}$ denotes an average over samples.

Since we expect strong finite size effects at the critical point, we calculated data for different system sizes. Data for the distribution of $-\ln S(\tau)$ for $L = 128$ and $L = 256$ is shown in Figs. 1 and 2. One observes that the distributions are broad and that for larger times the probability distribution gains more weight in the tail, indicating that correlations decrease for larger times, as expected.

Since the curves for fixed L and different times τ appear to be shifted by a roughly constant amount on the (logarithmic) x-axis, we attempt a scaling plot of the data with the parameter free scaling variable

$$\alpha = -\frac{\ln S(\tau)}{\ln \tau}\,. \tag{6}$$

Note that in Ref. 12, which investigated the dynamics in the paramagnetic phase, the scaling variable was found to be $-\ln S(\tau)/\tau^{1/\mu}$, where μ is expected to diverge at the critical point. Hence α in Eq. (6) is a natural scaling variable at the critical point.

The corresponding scaling plots for $P(\alpha)$ against α are shown in Figs. 3 and 4 for $L = 128$ and $L = 256$. One sees that the data collapse is good for α not too large and that the range of α where scaling works *increases*

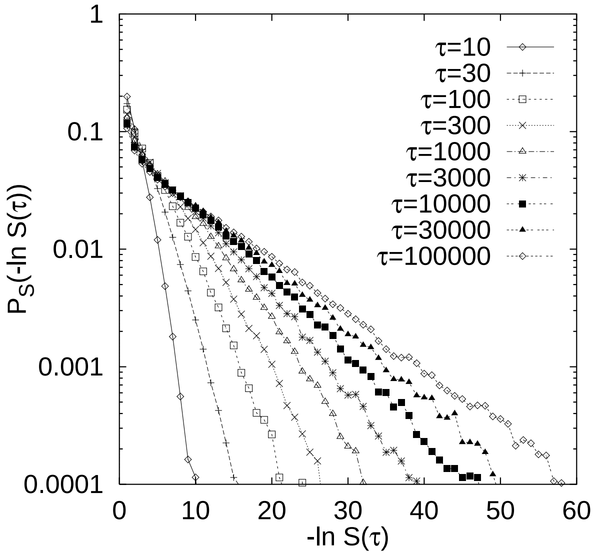

FIG. 2. The probability distribution of $-\ln S(\tau)$ for $L = 256$ at the critical point $\delta = 0$. The disorder average is over 10000 samples.

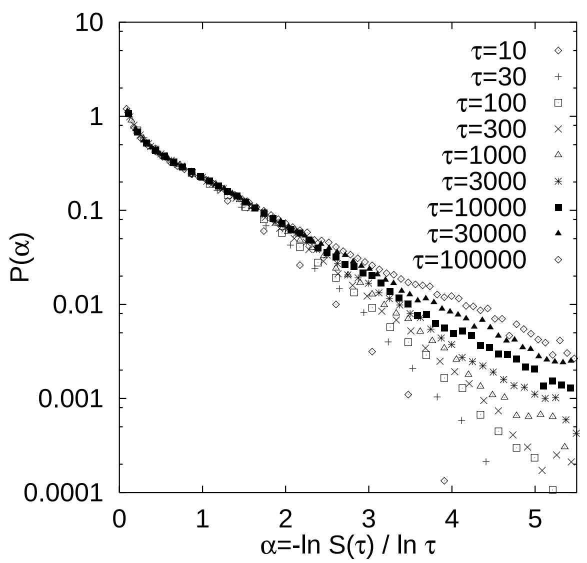

FIG. 3. Scaling plot of the probability distribution in Fig. 1 ($L = 128$). The scaling variable α is that given in Eq. (6). For larger values of α systematic deviations from scaling occur. This comes from data for small times τ which is presumably not in the scaling region.

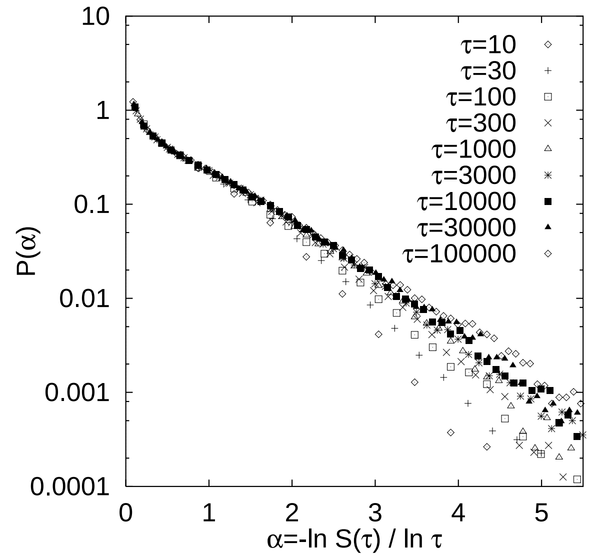

FIG. 4. Scaling plot of the probability distribution in Fig. 2 ($L = 256$). The scaling variable α is that given in Eq. (6). Note, by comparison with Fig. 3 that the range where the data scale well increases with increasing system size.

with increasing L. The data for large α where scaling breaks down corresponds to *small* τ, and it is reasonable to expect deviations from scaling in this region.

Since the data is only a function of the scaling variable α, it follows that typically the correlation function falls off with a power of τ. We shall now see that the precise value of the power depends in detail on how the typical correlation function is defined. For example, if we define "typical" to be the exponential of the average of the log, i.e.

$$S_{\rm avlog}(\tau) = \exp([\ln S(\tau)]_{\rm av})\,, \tag{7}$$

we obtain

$$[\ln S(\tau)]_{\rm av} = -\int_0^\infty P(\alpha)\alpha \ln\tau \, d\alpha = -\langle\alpha\rangle \ln\tau\,, \tag{8}$$

which yields, the algebraic decay

$$S_{\rm avlog}(\tau) = \tau^{-\langle\alpha\rangle}\,, \tag{9}$$

where $\langle \ldots \rangle$ denotes an average with respect to the distribution $P(\alpha)$. From our data we get $\langle\alpha\rangle \simeq 0.7$.

On the other hand if we define "typical" to be the median of the distribution, then one easily sees that

$$S_{\rm median}(\tau) = \tau^{-\alpha_{\rm med}}, \tag{10}$$

where $\alpha_{\rm med}$ is the median of $P(\alpha)$, i.e. it is defined implicitly by

$$\frac{1}{2} = \int_0^{\alpha_{\rm med}} P(\alpha)\, d\alpha. \tag{11}$$

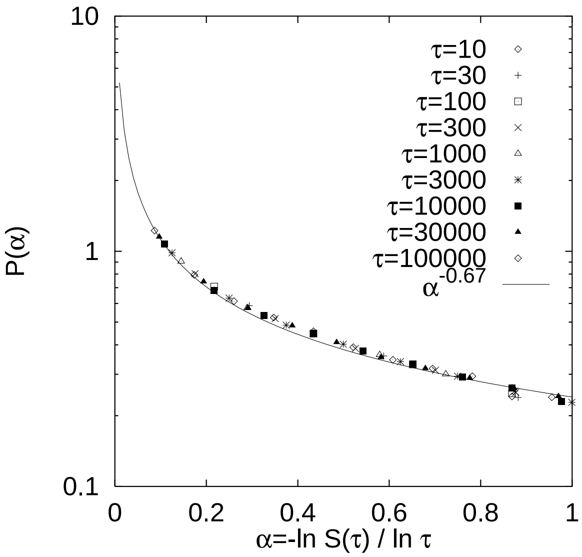

FIG. 5. An enlarged plot of the data in Fig. 4 (L=256). The full line is $\sim \alpha^{-0.67}$.

Any reasonable definition of "typical" will give a power law, in contrast to the average which has a much slower logarithmic variation, which we discuss next.

Contributions to the *average* correlation function can come both from the scaling function and from non-scaling contributions[14,10]. The scaling part comes from the small α part of the scaling function where there is an upturn in the data, similar to that found for the scaling function of the distribution of the static spin-spin correlations $C(r) = \langle \sigma_i^z(0)\sigma_{i+r}^z(0)\rangle$ at the critical point[14]. If we assume an algebraic relation $P(\alpha) \sim \alpha^{-\lambda}$ for small α, we can calculate the scaling contribution to the average correlation $S_{\rm av}(\tau)$ from

$$S_{\rm av}(\tau) = \int_0^\infty P(\alpha) S(\tau)\, d\alpha. \tag{12}$$

Noting that $S(\tau) = \exp(-\alpha \ln \tau)$ one obtains

$$\begin{aligned} S_{\rm av}(\tau) &\sim \int_0^\infty \alpha^{-\lambda} \exp(-\alpha \ln \tau)\, d\alpha \\ &\sim (\ln \tau)^{-(1-\lambda)}. \end{aligned} \tag{13}$$

This agrees with the results of Rieger and Iglói[13] who found

$$S_{\rm av}(\tau) \sim (\ln \tau)^{-2x_m}\,, \tag{14}$$

(where $x_m = (1-\phi/2) \simeq 0.191$ is the bulk magnetization exponent with $\phi = (1+\sqrt{5})/2$), provided $\lambda = 1 - 2x_m \simeq 0.618$.

To check this we show in Fig. 5 an enlarged plot of the data in Fig. 4 ($L = 256$) for small α, together with the function $\sim \alpha^{-0.67}$, which is the best power law fit to the data. The curve fits the data fairly well and the exponent of 0.67 is reasonably close to the value of $\lambda \simeq 0.618$ calculated above. Note again that there may be *additional* non-scaling contributions to the average correlation function as in Ref. 10.

III. CONCLUSIONS

We have studied numerically the distribution of local (on-site) correlations in imaginary time for the random transverse-field Ising chain at the critical point. The distribution was found to be logarithmically broad and the scaling variable $\alpha = \log S(\tau)/\log \tau$ was established. This means that while the correlations typically decay with a power of τ, there is a *range* of exponents α with a distribution $P(\alpha)$. The small α part of the scaling function, which dominates the average correlations, can be fitted by $P(\alpha) \sim \alpha^{-0.67}$, which gives (close to) the correct exponent in Eq. (13) for the logarithmic decay of the average correlation function. Note that *all* positive moments are determined by the small α region and so fall off with the *same*[25] decay given in Eq. (13).

The behavior of the distribution of $S(\tau)$ found here is somewhat analogous to "multifractal" behavior[19] predicted for the decay of *spatial* correlations in the *classical* two-dimensional Potts model at the critical point[20], since both have a distribution of scaling exponents. Beyond that, however, there are significant differences. Whereas in our case the probability of having a scaling exponent α is $P(\alpha)$, which does not explicitly depend on τ, for the corresponding multifractal behavior, the probability would be $\tau^{-f(\alpha)}$. As a result, for multifractal behavior, the typical correlation function has an exponent $\alpha_{\rm min}$, the value of α at the minimum of $f(\alpha)$, whereas here we find that the exponent for the typical correlation function depends on exactly how "typical" is defined. Averages of the n-th moment of the correlation function for positive n are also quite different. In our case, for all $n > 0$, the behavior is dominated by the small α region of the distribution and the exponent is independent of n[25], whereas for multifractal behavior, the moments depend on n in a non-trivial way and are given by[19,20] the Legendre transform of $f(\alpha)$. It would be interesting to see if there are other systems which have a "multiscaling" behavior of the type found here.

IV. ACKNOWLEDGMENTS

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